Int. J. Solids Structures Vol. 28, No. 1, pp. 17-32, 1991
Printed in Great Britain

0020-7683/91 $3.00+.00

FREE VIBRATION OF SHEAR-FLEXIBLE ANTI-SYMMETRIC ANGLE-PLY DOUBLY CURVED PANELS

HUMAYUN R. H. KABIR† and REAZ A. CHAUDHURI
Department of Civil Engineering, University of Utah, 3220 Merrill Engineering Building, Salt Lake City, UT 84112, U.S.A.

(*Received 3 January 1990; in revised form 18 August 1990*)

Abstract—A hitherto unavailable analytical solution to the boundary-value problem of free vibration of an anti-symmetric angle-ply laminated shear-flexible doubly curved shell of rectangular planform is presented. A novel solution methodology, based on a boundary-continuous double Fourier series approach, is developed to solve the eigenvalue problems, involving five highly coupled linear partial differential equations with constant coefficients, resulting from Sanders' FSDT (first-order shear-deformation theory)-based formulation that also includes surface-parallel and rotatory inertias. Numerical results presented in this study exhibit, for the first time, a mode switch of numerically ordered frequencies from transverse to surface-parallel modes with the change of such geometric and material parameters as length-to-thickness ratio, radius-to-thickness ratio and lamination angle. Additionally, these results have been utilized to validate the accuracy of available CLT-based approximate solutions, computed using the Galerkin approach.

1. INTRODUCTION

It is well established that analysis of laminated curved panels fabricated with such advanced composite materials as graphite/epoxy, boron/epoxy, graphite/PEEK etc., are complex due to their inherent in-plane anisotropy and asymmetry of lamination, resulting in various coupling effects, e.g. bending stretching coupling, first studied by Ambartsumyan (1953). Additional complexities arise because of transverse shear deformation, caused by low transverse shear modulus-to-in-plane Young's modulus ratio and effect of boundary constraints.

Stavsky and Lowey (1971), Jones and Morgan (1975) and Greenberg and Stavsky (1980) have all obtained exact solutions [in the sense that an infinite set of linear algebraic equations can be truncated to any desired degree of accuracy, according to Chia (1977); see also Chaudhuri and Abu-Arja (1991)] for the vibration and buckling problems of thin cross-ply cylindrical shells. Soldatos and Tzivanidis (1982) have presented exact solutions to the vibration and buckling problems of cross-ply cylindrical panels. Jones and Morgan (1975) and Soldatos and Tzivanidis (1982) have used Donnell's kinematic relations, while Stavsky and Lowey (1971) and Greenberg and Stavsky (1980) have utilized a Love-type theory. Dong *et al.* (1962) have also developed a theory of anisotropic thin shells, employing Donnell's shell theory, and presented results for cross-ply laminates. All of the aforementioned exact solutions for thin shells (based on Kirchhoff–Love's hypothesis where transverse shear deformations are neglected) are limited to: (1) cylindrical geometry, (2) cross-ply laminations, and (3) special boundary conditions termed SS3 [under the classification of Hoff and Rehfield (1965), which will be used in this paper henceforth]. Utilizing the approximate Galerkin approach, Soldatos (1982) has obtained solutions to the free vibration problems of anti-symmetric angle-ply thin cylindrical panels with SS2 boundary conditions.

Gulati and Essenberg (1967), and Zukas and Vinson (1971) have studied the effects of transverse shear deformation on the response of complete cylindrical shells, by introducing the first-order shear deformation theory (FSDT), based on the so-called Mindlin hypothesis. Dong and Tso (1972) have developed a FSDT-based theory for cross-ply shells and presented exact solutions for free vibration of complete cylindrical shells. Sinha and Rath (1976) have obtained exact solutions for transversely loaded circular cylindrical panels

† Presently, Senior Research Engineer, EMRC, Troy, MI 48083, U.S.A.

by incorporating the FSDT into Donnell's kinematic relations. Using four tracers to handle four popular shell theories due to, namely—Sanders (1959), Love, Morley (1959) and Donnell—Hsu *et al.* (1981) and Bert and Kumar (1982) have obtained FSDT-based exact solutions to thermal stress and vibration problems, respectively, of bimodulus cross-ply complete cylindrical shells and panels. Reddy (1984) has used Sanders' (1959) kinematic relations and the FSDT for solving the problems of bending and vibration of shear-flexible doubly curved panels. While he has been able to obtain exact solutions to the problems of cross-ply doubly curved panels with SS3-type simply supported boundary conditions, wherein the displacement functions have been expanded into double Fourier series, his attempt at obtaining an exact solution to the problem of anti-symmetric angle-ply doubly curved shells with SS2-type boundary conditions has not been crowned with success, because of his ending up with 10 sets of linear algebraic equations in five sets of unknowns, leading to his conclusion that "unlike plates, anti-symmetric angle-ply laminated shells with simply supported boundary conditions do not admit exact solutions". The central issue here is the well-posedness of the Fourier formulation, introduced through a Navier-type approach. The first objective of the present study is to devise a method that will ensure the well-posedness of the formulation, so that the number of equations becomes equal to the number of unknown Fourier coefficients to furnish a unique complete solution. Study of the effects of various geometric and material parameters on the computed natural frequencies of such laminated panels, which are extremely important design considerations, will comprise the second objective of this investigation. Furthermore, a literature search reveals that the effect of surface-parallel inertia terms, resulting in mode switch with the geometric and material parameters, is yet to be investigated, which will form the third objective of this paper. In addition, numerical results thus obtained will be compared to the available CLT-based results, computed using the approximate Galerkin approach due to Soldatos (1982), which will form the final objective of the present study.

2. STATEMENT OF THE PROBLEM

Figure 1 shows a laminated doubly curved panel (open shell) of rectangular planform, of total thickness h. x_1 and x_2 represent the directions of the lines of curvature of the middle surface, while the x_3-axis is a straight line perpendicular to the middle surface. R_i $(i = 1, 2)$ denotes the principal radii of curvature of the middle surface. The following set of simplifying assumptions is considered: (i) first-order shear deformation theory (FSDT); (ii) shallow shell approximation—$h/R_1, h/R_2 \ll 1$; (iii) transverse inextensibility; and (iv) neglect of the geodesic curvature.

The displacement fields, based on the above hypotheses, are:

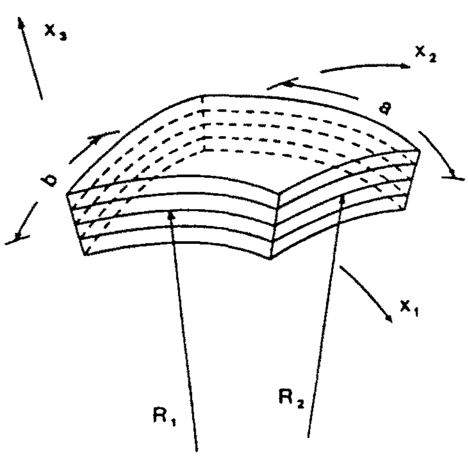

Fig. 1. A doubly curved panel.

$$u_i' = (1+x_3/R_i)u_i + x_3\phi_i; \quad i = 1,2; \quad u_3' = u_3 \tag{1}$$

in which u_i' $(i = 1, 2, 3)$ represents the components of displacement at a point x_i $(i = 1, 2, 3)$; and u_i denotes the same for the corresponding point at the midsurface. The strain displacement relations of a doubly curved shell are (Reddy, 1984):

$$\varepsilon_1 = \varepsilon_1^0 + x_3 k_1; \quad \varepsilon_2 = \varepsilon_2^0 + x_3 k_2; \quad \varepsilon_4 = \varepsilon_4^0; \quad \varepsilon_5 = \varepsilon_5^0; \quad \varepsilon_6 = \varepsilon_6^0 + x_3 k_6 \tag{2}$$

where

$$\varepsilon_1^0 = u_{1,1} + \frac{u_3}{R_1}; \quad \varepsilon_2^0 = u_{2,2} + \frac{u_3}{R_2}; \quad \varepsilon_4^0 = u_{3,2} + \phi_2 - \frac{u_2}{R_2}$$

$$\varepsilon_5^0 = u_{3,1} + \phi_1 - \frac{u_1}{R_1}; \quad \varepsilon_6^0 = u_{2,1} + u_{1,2}; \quad \kappa_1 = \phi_{1,1}$$

$$\kappa_2 = \phi_{2,2}; \quad \kappa_6 = \phi_{2,1} + \phi_{1,2} + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)(u_{2,1} - u_{1,2}) \tag{3}$$

in which ϕ_1 and ϕ_2 are the rotations of the reference surface (at $x_3 = 0$) about the x_2 and x_1 co-ordinate axes, respectively. The equations of motion, based on Sanders' (1959) shell theory, can be written as:

$$N_{1,1} + N_{6,1} + cM_{6,2} + \frac{Q_1}{R_1} = \left(P_1 + \frac{2P_2}{R_1}\right)u_{1,tt} + \left(P_2 + \frac{P_3}{R_1}\right)\phi_{1,tt}$$

$$N_{6,1} - cM_{6,1} + N_{2,2} + \frac{Q_2}{R_2} = \left(P_1 + \frac{2P_2}{R_2}\right)u_{2,tt} + \left(P_2 + \frac{2P_3}{R_2}\right)\phi_{2,tt}$$

$$Q_{1,1} + Q_{2,2} - \frac{N_1}{R_1} - \frac{N_2}{R_2} = P_1 u_{3,tt}$$

$$M_{1,1} + M_{6,2} - Q_1 = \left(P_2 + \frac{P_3}{R_1}\right)u_{1,tt} + (P_3)\phi_{1,tt}$$

$$M_{6,1} + M_{2,2} - Q_2 = \left(P_2 + \frac{P_3}{R_2}\right)u_{2,tt} + (P_3)\phi_{2,tt} \tag{4}$$

with

$$c = \frac{1}{R_1} - \frac{1}{R_2} \tag{5a}$$

$$(P_1, P_2, P_3) = \sum_{k=1}^{N} \int_{x_3^{(k-1)}}^{x_3^{(k)}} \rho^{(k)}(1, x_3, x_3^2)\, dx_3 \tag{5b}$$

where $\rho^{(k)}$ and N represent the density of the layer material and the total number of layers, respectively. N_1, N_2, N_6 are the surface-parallel stress resultants, while M_1, M_2, M_6 are moment resultants (stress couples), and Q_1 and Q_2 are the transverse shear-stress resultants, all per unit length. For an anti-symmetric angle-ply laminate,

$$A_{16} = A_{26} = A_{45} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0. \tag{6}$$

The stress and moment resultants are then defined as

$$N_1 = A_{11}\left(u_{1,1}+\frac{u_3}{R_1}\right)+A_{12}\left(u_{2,2}+\frac{u_3}{R_2}\right)+B_{16}[\phi_{2,1}+\phi_{1,2}-c(u_{2,1}-u_{1,2})]$$

$$N_2 = A_{12}\left(u_{1,1}+\frac{u_3}{R_1}\right)+A_{22}\left(u_{2,2}+\frac{u_3}{R_2}\right)+B_{26}[\phi_{2,1}+\phi_{1,2}-c(u_{2,1}-u_{1,2})]$$

$$N_6 = A_{66}(u_{2,1}+u_{1,2})+B_{16}\phi_{1,1}+B_{26}\phi_{2,2}$$

$$M_1 = B_{16}(u_{2,1}+u_{1,2})+D_{11}\phi_{1,1}+D_{12}\phi_{2,2}$$

$$M_2 = B_{26}(u_{2,1}+u_{1,2})+D_{12}\phi_{1,1}+D_{22}\phi_{2,2}$$

$$M_6 = B_{16}\left(u_{1,1}+\frac{u_3}{R_1}\right)+B_{26}\left(u_{2,2}+\frac{u_3}{R_2}\right)+D_{66}[\phi_{2,1}+\phi_{1,2}-c(u_{2,1}-u_{1,2})]$$

$$Q_1 = A_{55}\left(u_{3,1}+\phi_1-\frac{u_1}{R_1}\right)K_1^2$$

$$Q_2 = A_{44}\left(u_{3,2}+\phi_2-\frac{u_3}{R_2}\right)K_2^2 \tag{7}$$

in which A_{ij}, B_{ij} and D_{ij} $(i, j = 1, 2, 6)$ denote the extensional, flexural–extensional coupling, and flexural rigidities, while A_{ij}, $i, j = 4, 5$, are the transverse shear rigidities (Jones, 1975). K_1^2 and K_2^2 are the shear correction factors.

Substitution of eqns (7) into eqns (5) yields the following five highly coupled second-order partial differential equations:

$$\left(\frac{A_{55}}{R_1^2}\right)u_1+A_{11}u_{1,11}+2cB_{16}u_{1,12}+(A_{66}+c^2D_{66})u_{1,22}-cB_{16}u_{2,11}$$
$$+(A_{12}+A_{66}-c^2D_{66})u_{2,12}+cB_{26}u_{2,22}+\left(\frac{A_{11}}{R_1}+\frac{A_{12}}{R_2}+\frac{A_{55}}{R_1}\right)u_{3,1}+\left(\frac{cB_{16}}{R_1}+\frac{cB_{26}}{R_2}\right)u_{3,2}$$
$$+\left(\frac{A_{55}}{R_1}\right)\phi_1+2B_{16}\phi_{1,12}+cD_{66}\phi_{1,22}+B_{16}\phi_{2,11}+cD_{66}\phi_{2,12}$$
$$+B_{26}\phi_{2,22} = \left(P_1+\frac{2P_2}{R_1}\right)u_{1,tt}+\left(P_2+\frac{P_3}{R_1}\right)\phi_{1,tt} \tag{8a}$$

$$-cB_{16}u_{1,11}+(A_{66}+A_{12}-c^2D_{66})u_{1,12}+cB_{26}u_{1,22}+\left(\frac{A_{44}}{R_2^2}\right)u_2+(A_{66}+c^2D_{66})u_{2,11}$$
$$-2cB_{26}u_{2,12}+A_{22}u_{2,22}-c\left(\frac{B_{16}}{R_1}+\frac{B_{26}}{R_2}\right)u_{3,1}+\left(\frac{A_{12}}{R_1}+\frac{A_{22}}{R_2}+\frac{A_{44}}{R_2}\right)u_{3,2}+B_{16}\phi_{1,11}$$
$$-cD_{66}\phi_{1,12}+B_{26}\phi_{1,22}+\left(\frac{A_{44}}{R_2}\right)\phi_2-cD_{66}\phi_{2,11}$$
$$+2B_{26}\phi_{2,12} = \left(P_1+\frac{2P_2}{R_2}\right)u_{2,tt}+\left(P_2+\frac{P_3}{R_2}\right)\phi_{2,tt} \tag{8b}$$

$$\left(\frac{A_{55}}{R_1}-\frac{A_{11}}{R_1}-\frac{A_{12}}{R_2}\right)u_{1,1}-c\left(\frac{B_{16}}{R_1}+\frac{B_{26}}{R_2}\right)u_{1,2}+c\left(\frac{B_{16}}{R_1}+\frac{B_{26}}{R_2}\right)u_{2,1}-\left(\frac{A_{44}}{R_2}+\frac{A_{12}}{R_2}+\frac{A_{22}}{R_2}\right)u_{2,2}$$
$$+\left(\frac{A_{11}}{R_1^2}-\frac{2A_{12}}{R_1R_2}-\frac{A_{22}}{R_2^2}\right)u_3+A_{55}u_{3,11}+A_{44}u_{3,22}+A_{55}\phi_{1,1}-\left(\frac{B_{16}}{R_1}+\frac{B_{26}}{R_2}\right)\phi_{1,2}$$
$$-\left(\frac{B_{16}}{R_1}+\frac{B_{26}}{R_2}\right)\phi_{2,1}+A_{44}\phi_{2,2} = P_1u_{3,tt} \tag{8c}$$

$$\left(\frac{A_{55}}{R_1}\right)u_1+2B_{16}u_{1,12}+cD_{66}u_{1,22}+B_{16}u_{2,11}-cD_{66}u_{2,12}+B_{26}u_{2,22}$$

$$-A_{55}u_{3,1}+\left(\frac{B_{16}}{R_1}+\frac{B_{26}}{R_2}\right)u_{3,2}-A_{55}\phi_1+D_{11}\phi_{1,11}+D_{66}\phi_{1,22}$$

$$(D_{12}+D_{66})\phi_{2,12}=\left(P_2+\frac{P_3}{R_1}\right)u_{1,tt}+P_3\phi_{1,tt} \quad \text{(8d)}$$

$$B_{16}u_{1,11}+cD_{66}u_{1,12}+B_{26}u_{1,22}+\left(\frac{A_{44}}{R_2}\right)-cD_{66}u_{2,11}+2B_{26}u_{2,12}-A_{44}u_{3,2}$$

$$+\left(\frac{B_{16}}{R_1}+\frac{B_{26}}{R_2}\right)u_{3,1}+(D_{12}+D_{66})\phi_{1,12}-A_{44}\phi_2+D_{66}\phi_{2,11}$$

$$+(D_{22})\phi_{2,22}=\left(P_2+\frac{P_3}{R_2}\right)u_{2,tt}+P_3\phi_{2,tt}. \quad \text{(8e)}$$

SS2-type simply supported boundary conditions are given by

$$u_n=u_3=\phi_t=M_n=N_6=0, \quad \text{at the edges } x_n=\text{constant}\ ; \quad n=1,2 \quad (9)$$

where n and t denote the normal and tangential directions to an edge and when $n = 1$, $t = 2$ and vice versa.

3. THE SOLUTION TECHNIQUE

Reddy (1984) has sought to solve the boundary-value problem, represented by eqns (8), (9), by assuming the displacement functions in the following form:

$$u_1=\sum_m^{L}\sum_n^{L}U_{mn}\sin(\alpha_m x_1)\cos(\beta_n x_2)T \quad 0\leqslant x_1\leqslant a\ ; \quad 0\leqslant x_2\leqslant b$$

$$u_2=\sum_m^{L}\sum_n^{L}V_{mn}\cos(\alpha_m x_1)\sin(\beta_n x_2)T \quad 0\leqslant x_1\leqslant a\ ; \quad 0\leqslant x_2\leqslant b$$

$$u_3=\sum_m^{L}\sum_n^{L}W_{mn}\sin(\alpha_m x_1)\sin(\beta_n x_2)T \quad 0\leqslant x_1\leqslant a\ ; \quad 0\leqslant x_2\leqslant b$$

$$\phi_1=\sum_m^{L}\sum_n^{L}X_{mn}\cos(\alpha_m x_1)\sin(\beta_n x_2)T \quad 0\leqslant x_1\leqslant a\ ; \quad 0\leqslant x_2\leqslant b$$

$$\phi_2=\sum_m^{L}\sum_n^{L}Y_{mn}\sin(\alpha_m x_1)\cos(\beta_n x_2)T \quad 0\leqslant x_1\leqslant a\ ; \quad 0\leqslant x_2\leqslant b \quad (10)$$

with $T = e^{i\omega t}$ for free vibration, while α_m and β_n are equal to $m\pi/a$ and $n\pi/b$, respectively.

The above assumed displacement functions (10) completely satisfy the geometric and natural boundary conditions as stipulated in eqns (9) in a manner similar to Navier's approach. Hence, these functions are expected to be well behaved in the vicinity of an edge and their substitution into the differential equations should pose no difficulty (Hobson, 1926). By the introduction of eqns (10) into e.g. eqn (8a), Reddy (1984) has obtained equations similar to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(\alpha_m x_1)\cos(\beta_n x_2)[\{G(i,1)-G(i,2)\alpha_m^2-G(i,4)\beta_n^2\}\alpha_m\beta_n U_{mn}$$
$$-G(i,7)\alpha_m\beta_n V_{mn}+G(i,10)\beta W_{mn}-G(i,13)\alpha_m\beta_n X_{mn}+\{G(i,15)-G(i,6)\alpha_m^2$$
$$-G(i,8)\beta_n^2\}Y_{mn}]+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(\alpha_m x_1)\sin(\beta_n x_2)[-G(i,3)\alpha_m\beta_n U_{mn}$$
$$+\{G(i,5)-G(i,6)\alpha_m^2-G(i,8)\beta_n^2\}V_{mn}+G(i,9)\alpha_m W_{mn}+\{G(i,11)-G(i,12)\alpha_m^2$$
$$-G(i,14)\beta_n^2\}X_{mn}-G(i,7)\alpha_m\beta_n Y_{mn}] = C_i \quad (11)$$

wherein $i = 1$ and

$$C_1 = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(\alpha_m x_1)\cos(\beta_n x_2)\omega^2\left\{P_1+\frac{2P_2}{R_1}\right\}U_{mn}$$
$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(\alpha_m x_1)\sin(\beta_n x_2)\omega^2\left\{P_2+\frac{P_3}{R_2}\right\}X_{mn}. \quad (12)$$

Constants $G(i, j)$; $i = 1, \ldots, 5$; $j = 1, \ldots, 18$ are as presented in eqns (A1) of the Appendix. On setting the coefficients of $\sin(\alpha_m x_1)\cos(\beta_n x_2)$ and $\cos(\alpha_m x_1)\sin(\beta_n x_2)$, in eqn (11), to zero, Reddy (1984) has obtained two sets of linear algebraic equations. Using the same approach, the remaining four equations of eqns (8) furnish a further eight sets of linear algebraic equations, finally, yielding, in total, $10mn$ equations in $5mn$ unknowns, for $m, n = 1, 2, 3, \ldots$, which has prompted Reddy (1984) to conclude that "unlike plates, antisymmetric angle-ply laminated shells with simply supported boundary conditions do not admit exact solutions".

The first step in alleviating the difficulty, encountered by Navier's approach employed by Reddy (1984), comprises assuming that the displacement functions are in a form identical to eqns (10), except that the lower limits of the cosine series include the m or $n = 0$ terms, which appear to have been excluded in Reddy's (1984) assumed solutions. Substitution of these modified displacement functions into the eqns (8a), (8e) will yield:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(\alpha_m x_1)\cos(\beta_n x_2)[\{G(i,1)-G(i,2)\alpha_m^2-G(i,4)\beta_n^2\}\alpha_m\beta_n U_{mn}$$
$$-G(i,7)\alpha_m\beta_n V_{mn}+G(i,10)\beta_n W_{mn}-G(i,13)\alpha_m\beta_n X_{mn}+\{G(i,15)-G(i,6)\alpha_m^2$$
$$-G(i,8)\beta_n^2\}Y_{mn}]+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(\alpha_m x_1)\sin(\beta_n x_2)[-G(i,3)\alpha_m\beta_n U_{mn}$$
$$+\{G(i,5)-G(i,6)\alpha_m^2-G(i,8)\beta_n^2\}V_{mn}+G(i,9)\alpha_m W_{mn}+\{G(i,11)-G(i,12)\alpha_m^2$$
$$-G(i,14)\beta_n^2\}X_{mn}-G(i,7)\alpha_m\beta_n Y_{mn}]$$
$$+\sum_{m=1}^{\infty}\sin(\alpha_m x_1)[\{G(i,1)-G(i,2)\alpha_m^2\}U_{m0}+\{G(i,15)-G(i,16)\beta_n^2\}Y_{m0}]$$
$$+\sum_{n=1}^{\infty}\cos(\beta_n x_2)[\{G(i,5)-G(i,8)\beta_n^2\}V_{0n}+\{G(i,11)-G(i,14)\beta_n^2\}X_{0n}] = C_i$$
$$\text{for}\quad i = 1,5 \quad\text{and}\quad 0 < x_1 < a;\quad 0 < x_2 < b. \quad (13)$$

The next and more important step is to expand the $\cos(\alpha_m x_1)\sin(\beta_n x_2)$ and $\cos(\beta_n x_2)$ functions in the form of Fourier series, as suggested by Green and Hearmon (1945) as follows:

$$\cos(\alpha_m x_1)\sin(\beta_n x_2) = \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} h_{rm}h_{sn}\sin(\gamma_r x_1)\cos(\psi_s x_2) \quad 0 < x_1 < a;\quad 0 < x_2 < b \quad (14a)$$

and

$$\cos(\beta_n x_2) = \sum_{s=1}^{\infty} \sin(\psi_s x_2) \qquad 0 < x_2 < b \tag{14b}$$

where

$$h_{rm} = \frac{4m}{\pi(m^2-r^2)}, \quad h_{sn} = \frac{4n}{\pi(n^2-s^2)}; \quad m \neq r, \quad n \neq s \tag{15a}$$

and

$$\gamma_r = \pi r/a, \quad \psi_s = \pi s/b. \tag{15b}$$

Substituting the series expansions (14) into eqn (13) and equating to zero the coefficients of $\sin(\alpha_m x_1)\cos(\beta_n x_2)$ and $\sin(\alpha_m x_1)$ then furnishes, after a rearrangement,

$$\begin{aligned}
&\{G(i,1)-G(i,2)\alpha_m^2-G(i,4)\beta_n^2\}U_{mn}-G(i,7)\alpha_m\beta_n V_{mn}+G(i,10)\beta_n W_{mn} \\
&-G(i,3)\alpha_m\beta_n X_{mn}+\{G(i,15)-G(i,16)\alpha_m^2-G(i,18)\beta_n^2\}Y_{mn}+\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}h_{rm}h_{sn} \\
&\times\{G(i,3)\gamma_r\psi_s U_{rs}+\{G(i,5)-G(i,6)\gamma_r^2-G(i,8)\psi_s^2\}V_{rs}+G(i,9)\gamma_r W_{rs} \\
&+\{G(i,11)-G(i,12)\gamma_r^2-G(i,14)\psi_s^2\}X_{rs}-G(i,17)\gamma_r\psi_s Y_{rs}\}+h_{0m}\sum_{s=1}^{r}h_{sn} \\
&\times\{\{G(i,5)-G(i,8)\psi_s^2\}V_{0s}+\{G(i,11)-G(i,14)\psi_s^2\}X_{0s}\}=\bar{C}_i
\end{aligned} \tag{16}$$

$$\{G(i,1)-G(i,2)\alpha_m^2\}U_{m0}+\{G(i,15)-G(i,16)\alpha_m^2\}Y_{m0}=C_i' \tag{17}$$

in which $i = 1, 5$, while $\bar{C}_i$ and C_i' $(i = 1, \ldots, 5)$ are as defined by eqns (A2) and (A3), respectively, in the Appendix.

The last step has eliminated the difficulty encountered by Navier's approach employed by Reddy (1984) to solve the problem under consideration. One can now obtain, setting $i = 1, 5$ into the eqns (16), (17), two sets of linear algebraic equations corresponding to equations (8a), (8e). Similar operations on the remaining equations of eqns (8) will supply

$$\begin{aligned}
&-G(i,3)\alpha_m\beta_n U_{mn}+\{G(i,5)-G(i,6)\alpha_m^2-G(i,8)\beta_n^2\}V_{mn}+G(i,9)\alpha_m W_{mn} \\
&+\{G(i,11)-G(i,12)\alpha_m^2-G(i,14)\beta_n^2\}X_{mn}-G(i,17)\alpha_m\beta_n Y_{mn}+\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}h_{rm}h_{sn} \\
&\times\{\{G(i,1)-G(i,2)\gamma_r^2-G(i,4)\psi_s^2\}U_{rs}-G(i,7)\gamma_r\psi_s V_{rs}+G(i,10)\psi_s W_{rs}-G(i,13)\gamma_r\psi_s X_{rs} \\
&+\{G(i,15)-G(i,6)\gamma_r^2-G(i,8)\psi_s^2\}Y_{rs}\}+h_{0n}\sum_{r=1}^{\infty}h_{rm}[\{G(i,1)-G(i,2)\gamma_r^2\}U_{r0} \\
&+\{G(i,15)-G(i,16)\gamma_r^2\}]=\bar{C}_i \quad \text{for } i=2,4
\end{aligned} \tag{18}$$

$$\{G(i,5)-G(i,8)\beta_n^2\}V_{0n}+\{G(i,11)-G(i,14)\beta_n^2\}X_{0n}=C_i' \quad \text{in which } i=2,4 \tag{19}$$

and

$$\{-G(3,2)\beta_n U_{mn} - G(3,3)\alpha_m V_{mn}\} + \{G(3,5) - G(3,6)\alpha_m^2 - G(3,8)\beta_n^2\} W_{mn}$$

$$- G(3,9)\alpha_m X_{mn} - G(3,12)\beta_n Y_{mn} + \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} h_{rm}h_{sn}\{G(3,1)\gamma_r U_{rs} + G(3,4)\psi_s V_{rs}$$

$$+ G(3,7)\gamma_r\psi_s W_{rs} + G(3,10)\psi_s X_{rs} + G(3,11)\gamma_r Y_{rs}\} = \bar{C}_3. \quad (20)$$

Finally, for the free vibration problem under investigation, the above procedure has yielded a $5mn+2m+2n$ eigensystem (homogeneous linear algebraic equations in terms of as many unknowns), in which case the Fourier coefficients need not be determined explicitly. The eigenvalues and eigenvectors are computed by calling the software IMSL as a subroutine. The convergence characteristics of the Fourier series solution, which is an important issue in the case of static deformation problem, has been investigated by Kabir and Chaudhuri (1991) for the case of clamped cross-ply plate. Extension of the same to the case of a laminated doubly curved panel, the details of which are available in Kabir (1990), will be published in a follow-up paper.

4. RESULTS AND DISCUSSIONS

The following two examples—(i) cylindrical and (ii) spherical panels of square planform, which are special cases of doubly curved panels—will serve to illustrate the validity of the analytical procedure of the preceding section. Two examples of anti-symmetric angle-ply lamination—$(\theta/-\theta)$ and $(\theta/-\theta/\theta/-\theta)$—will be considered. Material properties are assumed identical to those of Soldatos (1982):

$$E_1/E_2 = 40; \quad G_{23}/E_2 = 0.50; \quad G_{12}/E_2 = G_{13}/E_2 = 0.6; \quad \nu_{12} = 0.25$$

wherein E_1 and E_2 are the Young's moduli in the directions parallel and transverse to the fibers, respectively. G_{12} is the surface-parallel shear modulus, while G_{13} and G_{23} are the transverse shear moduli and ν_{12} is the major Poisson's ratio. The shear correction factors considered, $K_1^2 = K_2^2 = 5/6$, are fairly standard. Normalized frequencies are defined as

$$\bar{\omega}_i = \omega_i a^2\sqrt{\rho/E_2}/h \quad \text{for} \quad i = 1,2,\ldots. \quad (21)$$

Example (i): Cylindrical panels of square planform

This example is selected for the purpose of verifying the convergence and also because CLT-based numerical results due to Soldatos (1982), computed using the approximate Galerkin's procedure, are available.

Table 1 shows the convergence of the fundamental frequency, ω_1, which corresponds to the mode shape $u_{3(1,1)}$, of a relatively flat $(R/a = 92.1403)$ and moderately-thick

Table 1. Comparison of convergence of the normalized fundamental frequency of a 45°/−45° cylindrical panel†

$n = m$	Soldatos (1982)	Present solution
1	27.120	27.019
2	26.354	27.262
3	26.298	27.375
4	26.171	27.420
5	26.166	27.415
6	—	27.430
7	—	27.440

† $E_1/E_2 = 40$; $G_{12}/E_2 = G_{13}/E_2 = 0.5$; $G_{23}/E_2 = 0.6$; $\nu_{12} = 0.25$; $R/a = 92.1403$; $a = b$.

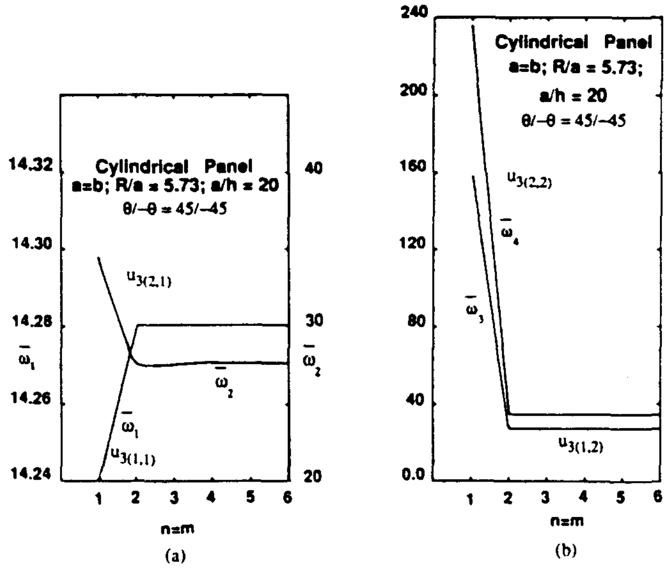

Fig. 2. Convergence of normalized (a) first and second, (b) third and fourth numerically ordered frequencies.

(a/h = 20) 45°/−45° cylindrical panel. Soldatos (1982), in his CLT-based study, has shown convergence up to $m, n = 5$, while up to $m, n = 7$ terms have been included in the present FSDT-based convergence study. Rapid convergence has been observed here, with results due to $m = n = 1$ being within less than 1.5% (error) of the converged solution due to $m = n = 7$. The approximate solution due to Soldatos (1982) is in close agreement with the present solution, the minor difference between the two solutions being attributable to Soldatos' use of the CLT and Galerkin's approach. The convergence of the first four natural frequencies is illustrated in Figs 2(a), (b) for a moderately-deep (R/a = 5.73) and moderately-thick (a/h = 20) 45°/−45° panel. Rapid convergence of the fundamental frequency for the latter-type panel confirms the same trend, demonstrated by Table 1, in the case of the former. The same trend continues for the second, third and fourth natural frequencies, wherein $m = n = 2$ terms appear to be adequate for numerical convergence, which renders the present analytical procedure also numerically efficient. Variation of the fundamental frequencies (ω_1), for moderately-thick (a/h = 20) and thick (a/h = 5) anti-symmetric angle-ply moderately-deep (R/a = 5.73) panels, as functions of lamination angle, θ, are presented in Figs 3(a) and 3(b), respectively. Each case considers two laminations—two-layer ($\theta/-\theta$) and four-layer ($\theta/-\theta/\theta/-\theta$). In the case of the thick four-layer ($\theta/-\theta/\theta/-\theta$) angle-ply panel, with $a/h = 5$, the maximum frequency is found to occur at a θ in the neighborhood of 60°, the minimum being at $\theta = 0°$. For a/h = 20 [Fig. 3(a)], ω_1 assumes its maximum and minimum values at $\theta = 90°$ and $\theta = 0°$, respectively. The two-layer anti-symmetric angle-ply panel assumes its minimum ω_1 near $\theta = 15°$, the maximum being at $\theta = 90°$ for both the thickness ranges—$a/h = 5$ and 20. Figure 4 shows the fundamental frequencies, ω_1, of two-layer ($\theta/-\theta$) and four-layer ($\theta/-\theta/\theta/-\theta$) moderately-thick ($a/h$ = 20) and deep (R/a = 1.30) panels as functions of the lamination angle, θ. The maximum frequency in both the cases occurs near $\theta = 82°$, while minimum values are assumed at different angles of lamination, with those for the two-layer and four-layer panels being close to 5° and at 20°, respectively. It is interesting to observe that for $\theta \geq 80°$ (approximate) a mode switch occurs, with the result that the fundamental frequency now corresponds to a surface-parallel mode, $u_{1(1,1)}$, instead of the usual transverse mode, $u_{3(1,1)}$.

Example (ii): Spherical panels of square planform

The second example considered is that of the simplest type of doubly curved panels, i.e. a spherical panel (wherein the geodesic curvature has been neglected), with the purpose

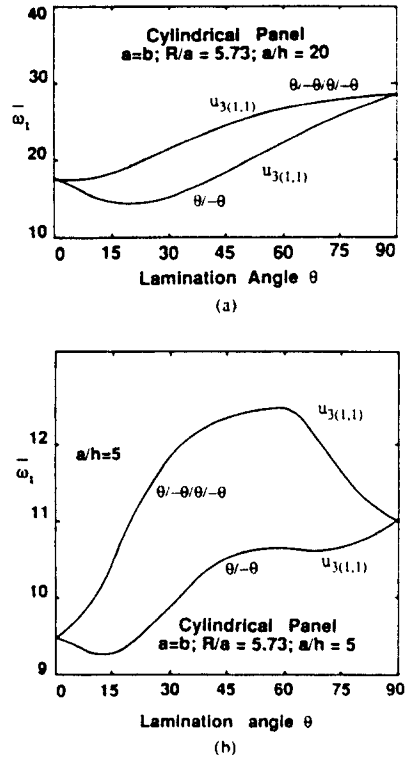

Fig. 3. Variation of normalized fundamental frequency with lamination angle for (a) moderately-thick and (b) thick, moderately-deep cylindrical panels.

of investigating the influence of the lamination angle, θ, curvature and thickness on the natural frequencies.

Figures 5(a) and 5(b) exhibit the variation of the fundamental frequency, ω_1, with respect to the lamination angle, θ, for moderately-thick ($a/h = 20$) and thick ($a/h = 5$) panels, respectively, with $R/a = 5.73$. In contrast to its cylindrical panel counterparts (Figs

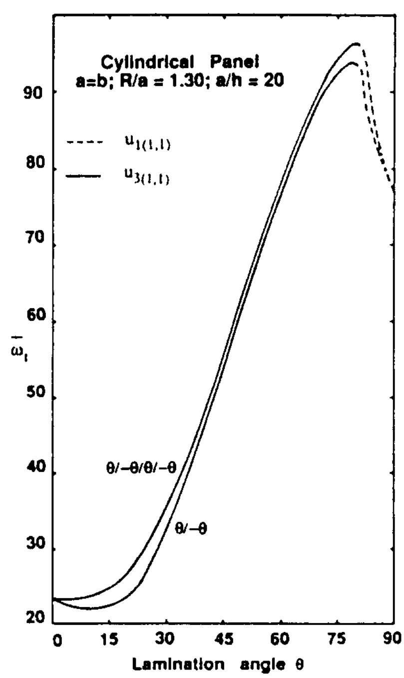

Fig. 4. Variation of the normalized fundamental frequency with lamination angle of moderately-thick deep cylindrical panels.

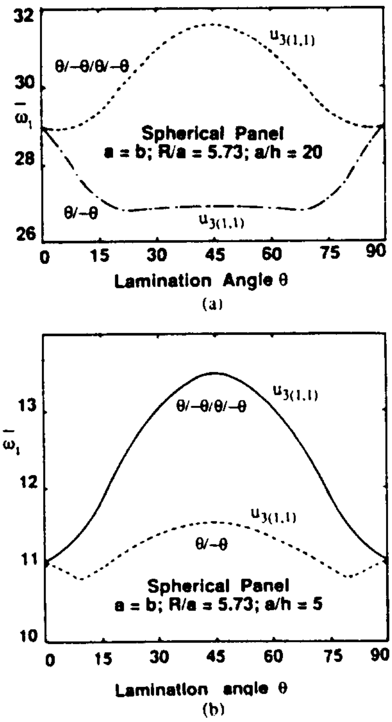

Fig. 5. Variation of normalized fundamental frequency with lamination angle for (a) moderately-thick and (b) thick, moderately-deep spherical panels.

3a, 3b), symmetry about $\theta = 45°$ of these plots (Figs 5a, 5b) is self-evident. Two-layer $\theta/-\theta$, moderately-thick ($a/h = 20$) panels attain maximum values at 0° and 90°, while four-layer, $\theta/-\theta/\theta/-\theta$, moderately-thick ($a/h = 20$) and both types of thick ($a/h = 5$) panels assume their maximum fundamental frequency values at $\theta = 45°$. Minimum values of $\bar{\omega}_1$, for a $\theta/-\theta$ panel with $a/h = 5$, occur close to $\theta = 10°$ and 80°, while their moderately-thick ($a/h = 20$) counterparts assume their minima at 21° and 69°. In the case of four-layer ($\theta/-\theta/\theta/-\theta$) panels, these minima occur at 0° and 90°, regardless of the shell thickness under consideration.

The influence of the curvature on the first four natural frequencies of moderately-thick ($a/h = 10$) two-layer (45°/−45°) and four-layer (45°/−45°/45°/−45°) panels is shown in Figs 6 and 7. The plot of $\bar{\omega}_1$ versus R/a (Fig. 6) demonstrates that for both 45°/−45° and

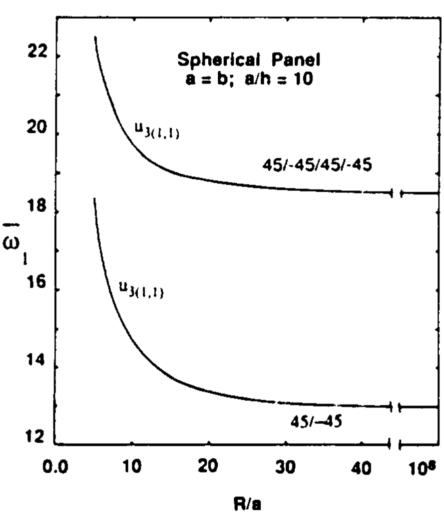

Fig. 6. Variation of the normalized fundamental frequency with R/a ratio, for moderately-thick spherical panels.

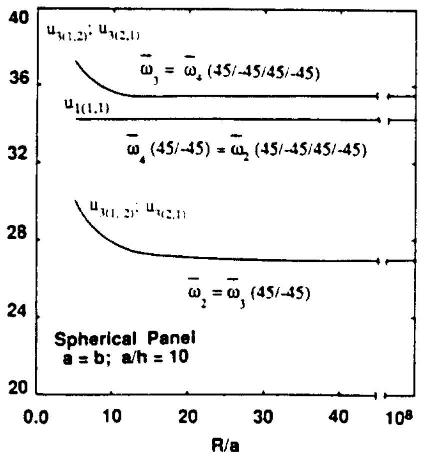

Fig. 7. Variation of the normalizd higher numerically-ordered frequencies with R/a ratio for moderately-thick spherical panels.

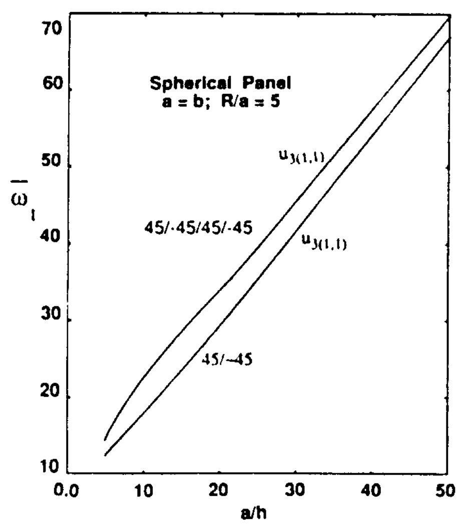

Fig. 8. Variation of normalized fundamental frequency with a/h ratio for moderately-deep spherical panels.

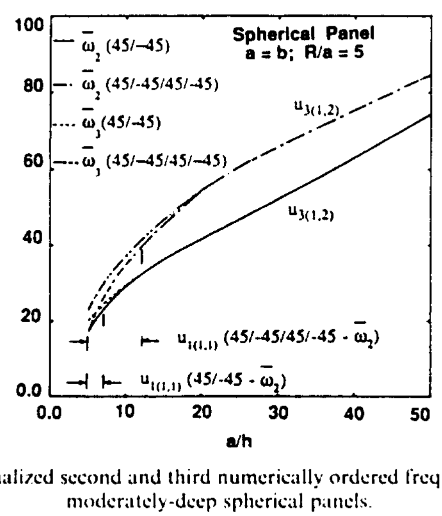

Fig. 9. Variation of normalized second and third numerically ordered frequencies with a/h ratio for moderately-deep spherical panels.

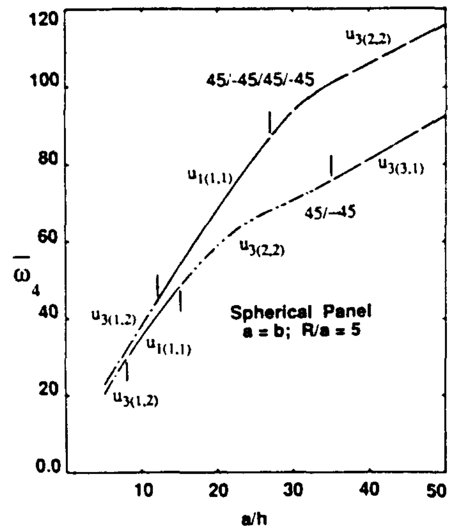

Fig. 10. Variation of the normalized fourth numerically ordered frequency with a/h ratio for moderately-deep spherical panels.

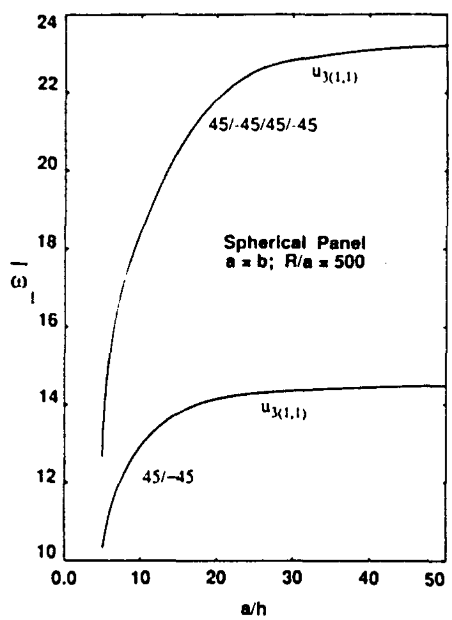

Fig. 11. Variation of normalized fundamental frequency with a/h ratio for very shallow spherical panels.

45°/−45°/45°/−45° laminates, the effect of shell curvature on ω_1 is not noticeable for $R/a \geqslant 30$ approximately. In the case of 45°/−45° laminates, ω_2 and ω_3, which correspond to the $u_{3(1,2)}$ and $u_{3(2,1)}$ modes, respectively, are almost identical (Fig. 7). This trend is also observed in the case of ω_3 and ω_4 of 45°/−45°/45°/−45° laminates, which correspond to the $u_{3(1,2)}$ and $u_{3(2,1)}$ modes, respectively. Further, it is interesting to observe a switch to a surface-parallel mode, $u_{1(1,1)}$, which corresponds to the numerical frequencies, ω_4, for 45°/−45° and ω_2 of 45°/−45°/45°/−45° laminates, which incidentally assume almost identical values for all R/a ratios.

The effect of thickness on the first four natural frequencies of two-layer, 45°/−45°, and four-layer, 45°/−45°/45°/−45°, moderately-deep panels ($R/a = 5$) is presented in Figs 8–10. Figure 9 exhibits a switch from out-of-plane or transverse mode, $u_{3(1,2)}$, to a surface-parallel mode, $u_{1(1,1)}$, in the case of the second (numerical) natural frequency, ω_2, for both 45°/−45° and 45°/−45°/45°/−45° laminates. Figure 9 also shows that ω_2 and ω_3 of

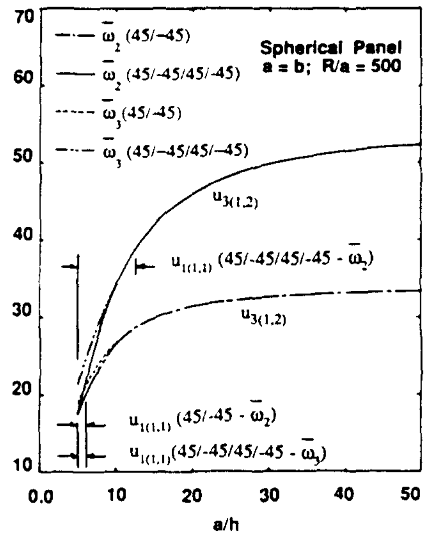

Fig. 12. Variation of normalized second and third numerically ordered frequencies with a/h ratio for very shallow spherical panels.

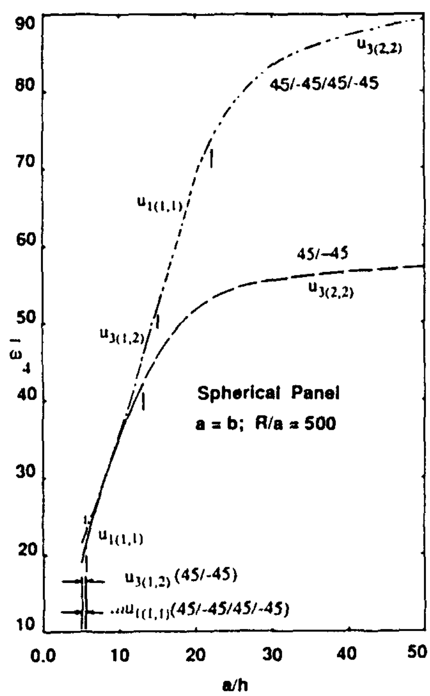

Fig. 13. Variation of the normalized fourth numerically ordered frequency with a/h ratio for very shallow spherical panels.

$45^\circ/-45^\circ$ merge together for $a/h \geq 10$ (approximately), while ω_2 and ω_3 of $45^\circ/-45^\circ/45^\circ/-45^\circ$ does the same for $a/h \geq 20$. Figure 10 shows the variation of the fourth numerical frequency, ω_4, with respect to the a/h ratio, for both $45^\circ/-45^\circ$ and $45^\circ/-45^\circ/45^\circ/-45^\circ$ laminates, wherein a switch of mode with the change in thickness is self-evident. Similar trends are also observed in the case of flatter panels (Figs 11–13). These plots further demonstrate that for relatively flat panels, nondimensionalized frequencies, $\bar{\omega}$, as expected, tend to become independent of the increase of a/h ratio, with the lower frequencies attaining this status faster than the higher ones.

5. CONCLUSIONS

A heretofore unavailable analytical solution to the problem of free vibration of a finite-dimensional anti-symmetric angle-ply shear-flexible doubly curved shell of rectangular planform is presented. Sanders' kinematic relations, extended to include the first-order shear deformation through the laminate thicknesss, and surface-parallel and rotatory inertias, are utilized in the formulation. A novel solution methodology, based on a boundary-continuous double Fourier series approach, is developed to solve the eigenvalue problems, involving five highly coupled linear partial differential equations with constant coefficients, previously thought to be incapable of admitting an "exact" solution. Numerical results presented here demonstrate fast convergence, and also testify to the accuracy and efficiency of the method developed. Furthermore, these results exhibit, for the first time, a mode switch of numerically ordered frequencies from transverse to surface-parallel modes with the change of such geometric and material parameters as the length-to-thickness ratio, radius-to-thickness ratio and lamination angle, which has profound implications for the role of these parameters on the failure modes of the type of composite panels under investigation. In addition, these results have been utilized to validate the accuracy of available CLT-based solutions, computed by using the approximate Galerkin approach. These solutions should serve as baselines for future comparison of results, obtained by such popular approximate numerical methods as finite elements and finite difference.

Acknowledgements—The first author's research was supported by the University of Utah Research Committee through a graduate research fellowship. The authors also wish to thank Prof. Steele, Editor, and the reviewers for their helpful comments on an earlier version of the manuscript.

REFERENCES

Ambartsumayan, S. A. (1953). Calculation of laminated isotropic shells. *Izv. Akad. Nauk Armen. SSR, Ser. Fiz. Matem. Eston. Techn. Nauk* **6**, 15.

Bert, C. W. and Kumar, M. (1982). Vibration of cylindrical shells of bimodulus composite materials. *J. Sound Vibr.* **81**, 107–121.

Chaudhuri, R. A. and Abu-Arja, K. R. (1991). Static analysis of moderately-thick finite antisymmetric angle-ply cylindrical panels and shells. *Int. J. Solids Structures* **28**, 1–15.

Chia, C. Y. (1977). *Nonlinear Analysis of Plates*. McGraw-Hill, New York.

Dong, S. B., Pister, K. and Taylor, R. C. (1962). On the theory of laminated anisotropic shells and plates. *J. Aero. Sci.* **29**, 969–975.

Dong, S. B. and Tso, F. K. W. (1972). On a laminated orthotropic shell theory including transverse shear deformation. *ASME J. Appl. Mech.* **39**, 1091–1096.

Green, A. E. and Hearmon, R. F. S. (1945). The buckling of flat rectangular plywood plates. *Phil. Mag.* **36**, 659–687.

Greenberg, J. B. and Stavsky, Y. (1980). Buckling and vibration of orthotropic composite cylindrical shells. *Acta Mech.* **36**, 15–29.

Gulati, S. T. and Essenberg, F. (1967). Effects of anisotropy in axisymmetric cylindrical shells. *ASME J. Appl. Mech.* **34**, 650–666.

Hobson, E. W. (1926). *The Theory of Functions of a Real Variable and the Theory of Fourier Series*, Vol. II, 2nd edn. Cambridge University Press, U.K.

Hoff, N. J. and Rehfield, L. W. (1965). Buckling of axially compressed circular cylindrical shells at stresses smaller than the classical critical value. *ASME J. Appl. Mech.* **32**, 542–546.

Hsu, Y. S., Reddy, J. N. and Bert, C. W. (1981). Thermoelasticity of cylindrical shells laminated of bimodulus composite materials. *J. Therm. Str.* **4**, 155–177.

Jones, R. M. (1975). *Mechanics of Composite Materials*. Scripta, Washington.

Jones, R. M. and Morgan, H. S. (1975). Buckling and vibration of cross-ply laminated circular cylindrical shells. *AIAA Jl* **13**, 664–671.

Kabir, H. R. H. (1990). Static and dynamic analysis of laminated finite doubly curved shells. Ph.D. Dissertation, University of Utah.

Kabir, H. R. H. and Chaudhuri, R. A. (1991). A generalized Navier's approach for solution of clamped moderately thick cross-ply plates. *Compos. Struct.* (in press).

Morely, L. S. D. (1959). An improvement of Donnell's approximation of thin-walled circular cylinders. *Q. J. Mech. Appl Maths.* **8**, 89–99.

Reddy, J. N. (1984). Exact solutions of moderately thick laminated shells. *J. Engng Mech. ASCE* **110**, 794–809.

Sanders, J. L. (1959). An improved first-approximation theory for thin shells. NASA Technical Report, R-24.

Sinha, P. K. and Rath, A. K. (1976). Transverse bending of cross-ply laminated circular cylindrical plates. *J. Mech. Engng Sci.* **18**, 53–56.

Soldatos, K. P. (1982). Free vibrations of antisymmetric angle-ply laminated circular cylindrical panels. *Q. J. Mech. Appl. Math.* **35**, 207–221.

Soldatos, K. P. and Tzivanidis, G. J. (1982). Buckling and vibration of cross-ply laminated circular cylindrical panels. *J. Appl. Maths. Phys. (ZAMP)* **33**, 230–239.

Stavsky, Y. and Lowey, R. (1971). On vibrations of heterogeneous orthotropic cylindrical shells. *J. Sound Vibr.* **15**, 235–256.

Zukas, J. A. and Vinson, J. R. (1971). Laminated transversely isotropic cylindrical shells. *ASME J. Appl. Mech.* **38**, 400–407.

APPENDIX

The non-zero constants, $G(i, j)$ with $i = 1$-5 and $j = 1$-18, are as given below:

$$G(1,1) = -\frac{A_{55}}{R_1^2};\quad G(1,2) = A_{11};\quad G(1,3) = 2cB_{16};\quad G(1,4) = A_{66}+c^2D_{66};$$

$$G(1,6) = -cB_{16};\quad G(1,7) = A_{12}+A_{66}-c^2D_{66};\quad G(1,8) = cB_{26};$$

$$G(1,9) = \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} + \frac{A_{55}}{R_1};\quad G(1,10) = \frac{cB_{16}}{R_1} + \frac{cB_{26}}{R_2};$$

$$G(1,11) = \frac{A_{55}}{R_1};\quad G(1,13) = 2B_{16};\quad G(1,14) = cD_{66};\quad G(1,16) = B_{16};$$

$$G(1,17) = cD_{66};\quad G(1,18) = B_{26};\quad G(2,2) = G(1,6);\quad G(2,3) = G(1,7);$$

$$G(2,4) = G(1,8);\quad G(2,5) = -\frac{A_{44}}{R_2^2};\quad G(2,6) = A_{66}+cD_{66};\quad G(2,7) = 2cB_{26};$$

$$G(2,8) = A_{22};\quad G(2,9) = -\frac{cB_{16}}{R_1} - \frac{cB_{26}}{R_2};\quad G(2,10) = \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} + \frac{A_{44}}{R_2};$$

$$G(2,12) = B_{16};\quad G(2,13) = -cD_{66};\quad G(2,14) = B_{26};$$

$$G(2,15) = \frac{A_{44}}{R_2^2};\quad G(2,16) = cD_{66};\quad G(2,17) = 2B_{26};\quad G(2,18) = B_{22};$$

$$G(3,1) = -\frac{A_{55}}{R_1} - \frac{A_{11}}{R_1} - \frac{A_{12}}{R_2};\quad G(3,2) = -\frac{cB_{16}}{R_1} - \frac{cB_{26}}{R_2};\quad G(3,3) = -G(3,2);$$

$$G(3,4) = -\frac{A_{44}}{R_2} - \frac{A_{12}}{R_1} - \frac{A_{22}}{R_2};\quad G(3,5) = -\frac{A_{11}}{R_1^2} - \frac{2A_{12}}{R_1R_2} - \frac{A_{22}}{R_2^2};\quad G(3,6) = A_{55};$$

$$G(3,8) = A_{44};\quad G(3,9) = A_{55};\quad G(3,10) = G(3,11) = G(3,2)/c;$$

$$G(3,12) = A_{44};\quad G(4,1) = \frac{A_{55}}{R_1};\quad G(4,3) = 2B_{16};\quad G(4,4) = cD_{66};\quad G(4,6) = B_{16};$$

$$G(4,7) = -cD_{66};\quad G(4,8) = B_{26};\quad G(4,9) = -A_{55};\quad G(4,10) = \frac{B_{16}}{R_1} + \frac{B_{26}}{R_2};$$

$$G(4,11) = -A_{55};\quad G(4,12) = D_{11};\quad G(4,14) = D_{66};\quad G(4,17) = D_{12}+D_{66};$$

$$G(5,2) = B_{16};\quad G(5,3) = cD_{66};\quad G(5,4) = B_{26};\quad G(5,5) = \frac{A_{44}}{R_2};\quad G(5,6) = -cD_{66};$$

$$G(5,7) = 2B_{26};\quad G(5,9) = -G(3,10);\quad G(5,10) = -A_{44};\quad G(5,13) = D_{66}+D_{12};$$

$$G(5,15) = -A_{44};\quad G(5,16) = D_{66};\quad G(5,18) = D_{22} \tag{A1}$$

$$C_1 = \sum_{m=1}^{t}\sum_{n=1}^{t} e^{i\omega t}\left[\left(P_1 + \frac{2P_2}{R_1}\right)U_{mn} + \sum_{r=1}^{t}\sum_{s=0}^{t} h_{rm}h_{sn}\left(P_2 + \frac{P_3}{R_1}\right)X_{rs}\right]\omega^2$$

$$C_2 = \sum_{m=1}^{t}\sum_{n=1}^{t} e^{i\omega t}\left[\left(P_1 + \frac{2P_2}{R_2}\right)V_{mn} + \sum_{r=1}^{t}\sum_{s=0}^{t} h_{rm}h_{sn}\left(P_2 + \frac{P_3}{R_2}\right)Y_{rs}\right]\omega^2$$

$$C_3 = -\sum_{m=1}^{t}\sum_{n=1}^{t} e^{i\omega t} P_1\omega^2$$

$$C_4 = -\sum_{m=1}^{t}\sum_{n=1}^{t} e^{i\omega t}\left[\sum_{r=0}^{t}\sum_{s=1}^{t} h_{rm}h_{sn}\left(P_2 + \frac{P_3}{R_1}\right) + P_3X_{mn}\right]\omega^2$$

$$C_5 = -\sum_{m=1}^{t}\sum_{n=1}^{t} e^{i\omega t}\left[\sum_{r=1}^{t}\sum_{s=0}^{t} h_{rm}h_{sn}\left(P_2 + \frac{P_3}{R_1}\right)V_{mn} + (P_3)Y_{mn}\right]\omega^2 \tag{A2}$$

$$C'_1 = -\sum_{m=1}^{t} e^{i\omega t}\left(P_1 + \frac{2P_2}{R_1}\right)\omega^2U_{m0}$$

$$C'_2 = -\sum_{n=1}^{t} e^{i\omega t}\left(P_1 + \frac{2P_2}{R_2}\right)\omega^2V_{0n}$$

$$C'_3 = 0$$

$$C'_4 = -\sum_{n=1}^{t} P_3\omega^2X_{0n}$$

$$C'_5 = -\sum_{m=1}^{t} P_3\omega^2Y_{m0}. \tag{A3}$$